Assignment 6.

This homework is due *Thursday*, October 15.

There are total 25 points in this assignment. 22 points is considered 100%. If you go over 22 points, you will get over 100% (up to 115%) for this homework and it will count towards your course grade.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper and *credit your collaborators*. Your solutions should contain full proofs. Bare answers will not earn you much.

This assignment covers sections 3.3–3.4 in Bartle–Sherbert.

(1) [3pt] (3.3.2) Let $x_1 > 1$ and $x_{n+1} = 2 - 1/x_n$ for $n \in \mathbb{N}$. Show that (x_n) is bounded and monotone, hence convergent. Find the limit.

(2) [2pt] Find a mistake in the following argument:
"Let (x_n) be a sequence given by $x_1 = 1$, $x_{n+1} = 1 - x_n$. In other words, $(x_n) = (1, 0, 1, 0, 1, 0, \dots)$. Show that $\lim(x_n) = 0.5$. Indeed, let $\lim(x_n) = x$. Apply limit to both sides of equality $x_{n+1} = 1 - x_n$:

$$\lim(x_{n+1}) = \lim(1 - x_n)$$
$$\lim(x_{n+1}) = 1 - \lim(x_n)$$
$$x = 1 - x,$$

so $x = 0.5$"

(3) [3pt] (3.3.11) Establish convergence or divergence of the sequence (y_n), where

$$y_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \qquad \text{for } n \in \mathbb{N}.$$

(4) (a) [2pt] (Exercise 3.3.12) Let $x_n = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}$, $n \in \mathbb{N}$. Prove that that (x_n) converges. (*Hint*: for $k \geq 2$, $\frac{1}{k^2} \leq \frac{1}{k(k-1)} = \frac{1}{k-1} - \frac{1}{k}$.)

(b) [2pt] Let K be a natural number $K \geq 2$. Let $y_n = \frac{1}{1^K} + \frac{1}{2^K} + \frac{1}{3^K} + \cdots + \frac{1}{n^K}$, $n \in \mathbb{N}$. Prove that that (y_n) converges. (*Hint*: compare[1] y_n to x_n.)

(5) (13.3.13abd) Establish the convergence and find the limits of the following sequences.
(a) [1pt] $((1 + 1/n)^{n+1})$,
(b) [1pt] $((1 + 1/n)^{-2n})$,
(c) [1pt] $((1 - 1/n)^n)$.
(*Hint:* Express these sequences through $X = ((1 + 1/n)^n)$. Use arithmetic properties of limit..)

— *see next page* —

[1] *Compare* here does not mean "write a short essay about how y_n is the same as x_n but with K", but rather determine which is greater.

(6) [2pt] (3.4.1) Give an example of an unbounded sequence that has a convergent subsequence.

(7) [3pt] (3.4.14) Let (x_n) be a bounded sequence and let

$$s = \sup\{x_n : n \in \mathbb{N}\}.$$

Show that if $s \notin \{x_n : n \in \mathbb{N}\}$, then there is a subsequence of (x_n) that converges to x.

(8) (a) [3pt] (3.4.9) Suppose that every subsequence of $X = (x_n)$ has a subsequence that converges to 0. Show that $\lim X = 0$.

(b) [2pt] Suppose that every subsequence of $X = (x_n)$ has a converging subsequence. Is it true that in this case, X must converge?